7 Topic: π is irrational

The goal of this project is to fill in the gaps in a relatively simple proof of the fact that π is irrational. Opposite to the previous projects, I will collect only one work from each group and each group member will get the same grade.

The idea of the proof is as usual: Assume that

$$\pi = \frac{a}{b}\,,$$

for positive integers a and b and reach a contradiction.

♢ **7.1.** The main tool in the proof is the function

$$f(x) = \frac{x^n(a-bx)^n}{n!}\,,$$

which is a polynomial of degree $2n$, where n is a natural number that will be specified later.

In the following you will need to compute derivatives of f. Note that f can be written as $f(x) = (n!)^{-1}u(x)v(x)$, where $u(x) = x^n$ and $v(x) = (a-bx)^n$. I hope that you remember how to find the derivative of a product. Continue taking the derivatives $(uv)', (uv)'', (uv)'''$, observe the pattern and realize how to compute $(uv)^{(k)}$, i.e., the k-th derivative of a product. (Hint: Project 3 can be of some help. The formula you obtain is often called the Leibnitz formula. Note that I do not request a proof of it.)

♢ **7.2.** Using the previous carefully prove

Lemma 7.1. *Assuming that $\pi = \frac{a}{b}$ and for f defined above the values*

$$f^{(k)}(0) \qquad f^{(k)}(\pi)$$

are integers for any integer $k \geq 0$.

Hint: The proof is somewhat tedious since you are literally asked to compute all $u^{(k)}(0)$, $u^{(k)}(\pi)$, $v^{(k)}(0)$, $v^{(k)}(\pi)$ (note that you will need to distinguish different cases for k, which ones?), and use the Leibnitz formula. Just do not skip any steps.

♢ **7.3.** Define two more functions:

$$\begin{aligned} g(x) &= f(x) - f''(x) + f^{(4)}(x) - \ldots + (-1)^n f^{(2n)}(x), \\ h(x) &= g'(x)\sin x - g(x)\cos x. \end{aligned}$$

Prove

Lemma 7.2.

$$h'(x) = f(x)\sin x.$$

(This lemma shows that $h(x)$ is an antiderivative of $f(x)\sin x$. The proof is a direct calculation of derivatives and canceling.)

$\diamondsuit$ **7.4.** Using Lemma 7.2 and 7.1 conclude that the integral

$$\int_0^\pi f(x) \sin x \mathrm{d}x$$

must be a positive integer.

On another hand justify (Calc I notes for basic properties of the integrals can be useful) the estimate

$$0 < \int_0^\pi f(x) \sin x \mathrm{d}x < \pi \frac{(\pi a)^n}{n!}.$$

Moreover, using any Calculus technique, show that

$$\frac{(\pi a)^n}{n!} \to 0$$

as n grows, and therefore it is always possible to choose n such that the integral above is (include what is needed here) hence reaching the contradiction.

$\diamondsuit$ **7.5.** Google the paper "A simple proof that π is irrational" by Ivan Niven, read it (it is half a page long) and test yourself whether you understand every step in the proof.